

ON INTERVALS $(kn, (k+1)n)$ CONTAINING A PRIME FOR ALL $n > 1$

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ABSTRACT. We study values of k for which the interval $(kn, (k+1)n)$ contains a prime for every $n > 1$. We prove that the list of such integers k includes $k = 1, 2, 3, 5, 9, 14$, and no others, at least for $k \leq 50,000,000$. For every known k of this list, we give a good upper estimate of the smallest $N_k(m)$, such that, if $n \geq N_k(m)$, then the interval $(kn, (k+1)n)$ contains at least m primes.

1. INTRODUCTION AND MAIN RESULTS

In 1850, P. L. Chebyshev proved the famous Bertrand postulate (1845) that every interval $[n, 2n]$ contains a prime (for a very elegant version of his proof, see Theorem 9.2 in [10]). Other nice proofs were given by S. Ramamujan in 1919 [8] and P. Erdős in 1932 (reproduced in [4], pp.171-173). In 2006, M. El. Bachraoui [1] proved that every interval $[2n, 3n]$ contains a prime, while A. Loo [6] proved the same statement for every interval $[3n, 4n]$. Moreover, A. Loo found a lower estimate for the number of primes in the interval $[3n, 4n]$. Note also that already in 1952 J. Nagura [7] proved that, for $n \geq 25$, there is always a prime between n and $\frac{6}{5}n$. From his result it follows that the interval $[5n, 6n]$ always contains a prime. In this paper we prove the following.

Theorem 1. *The list of integers k for which every interval $(kn, (k+1)n)$, $n > 1$, contains a prime includes $k = 1, 2, 3, 5, 9, 14$ and no others, at least for $k \leq 50,000,000$.*

Besides, in this paper, for every $k = 1, 2, 3, 5, 9, 14$, we give an algorithm for finding the smallest $N_k(m)$, such that, for $n \geq N_k(m)$, the interval $(kn, (k+1)n)$ contains at least m primes.

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2. CASE $k = 1$

Ramanujan [8] not only proved Bertrand's postulate but also indicated the smallest integers $\{R(m)\}$, such that, if $x \geq R(m)$, then the interval $(\frac{x}{2}, x]$ contains at least m primes, or, the same, $\pi(x) - \pi(x/2) \geq m$. It is easy to see that here it is sufficient to consider *integer* x and it is evident that every term of $\{R(m)\}$ is prime. The numbers $\{R(m)\}$ are called *Ramanujan primes* [14]. It is the sequence (A104272 in [13]):

$$(1) \quad 2, 11, 17, 29, 41, 47, 59, 67, 71, 97, \dots$$

Since $\pi(x) - \pi(x/2)$ is not a monotonic function, to calculate the Ramanujan numbers one should have an effective upper estimate of $R(m)$. In [8] Ramanujan showed that

$$(2) \quad \pi(x) - \pi(x/2) > \frac{1}{\ln x} \left(\frac{x}{6} - 3\sqrt{x} \right), \quad x > 300.$$

In particular, for $x \geq 324$, the left hand side is positive and thus ≥ 1 . Using direct descent, he found that $\pi(x) - \pi(x/2) \geq 1$ already from $x \geq 2$. Thus $R(1) = 2$ which proves the Bertrand postulate. Further, e.g., for $x \geq 400$, the left hand side of (2) is more than 1 and thus ≥ 2 . Again, using direct descent, he found that $\pi(x) - \pi(x/2) \geq 2$ already from $x \geq 11$. Thus $R(2) = 11$, etc. Sondow [14] found that $R(m) < 4m \ln(4m)$ and conjectured that

$$(3) \quad R(m) < p_{3m}$$

which was proved by Laishram [5]. Since, for $n \geq 2$, $p_n \leq en \ln n$ (cf. [3], Section 4), then (3) yields $R(m) \leq 3em \ln(3m)$, $m \geq 1$. Set $x = 2n$. Then, if $2n \geq R(m)$, then $\pi(2n) - \pi(n) \geq m$. Thus the interval $(n, 2n)$ contains at least m primes, if

$$n \geq \left\lceil \frac{R(m) + 1}{2} \right\rceil = \begin{cases} 2, & \text{if } m = 1, \\ \frac{R(m)+1}{2}, & \text{if } m \geq 2. \end{cases}$$

Denote by $N_1(m)$ the smallest number such that, if $n \geq N_1(m)$, then the interval $(n, 2n)$ contains at least m primes. It is clear, that $N_1(1) = R(1) = 2$. If $m \geq 2$, formally the condition $x = 2n \geq 2N_1(m)$ is not stronger than the condition $x \geq R(m)$, since the latter holds for x even and odd. Therefore, for $m \geq 2$, we have $N_1(m) \leq \frac{R(m)+1}{2}$. Let us show that in fact we have here the equality.

Proposition 2. For $m \geq 2$,

$$(4) \quad N_1(m) = \frac{R(m) + 1}{2}.$$

Proof. Note that the interval $\left(\frac{R(m)-1}{2}, R(m) - 1\right)$ cannot contain more than $m-1$ primes. Indeed, it is an interval of type $\left(\frac{x}{2}, x\right)$ for integer x and the following such interval is $\left(\frac{R(m)}{2}, R(m)\right)$. By the definition, $R(m)$ is the *smallest* number such that if $x \geq R(m)$, then $\left\{\left(\frac{x}{2}, x\right)\right\}$ contains $\geq m$ primes. Therefore, the supposition that already interval $\left(\frac{R(m)-1}{2}, R(m) - 1\right)$ contains $\geq m$ primes contradicts the minimality of $R(m)$. Since the following interval of type $(y, 2y)$ with integer $y \geq \frac{R(m)-1}{2}$ is $\left(\frac{R(m)+1}{2}, R(m) + 1\right)$, then (4) follows. \square

So the sequence $\{N_1(m)\}$, by (1), is (A084140 in [13])

$$(5) \quad 2, 6, 9, 15, 21, 24, 30, 34, 36, 49, \dots$$

3. GENERALIZED RAMANUJAN NUMBERS

Further our research is based on a generalization of Ramanujan's method. With this aim, we define generalized Ramanujan numbers (cf. [12], Section 10, and earlier (2009) comment in A164952 [13]).

Definition 3. Let $v > 1$ be a real number. A v -Ramanujan number $(R_v(m))$, is the smallest integer such that if $x \geq R_v(m)$, then $\pi(x) - \pi(x/v) \geq m$.

It is known [10] that all v -Ramanujan numbers are primes. In particular, $R_2(m) = R(m)$, $m = 1, 2, \dots$, are the proper Ramanujan primes.

Definition 4. For a real number $v > 1$ the v -Chebyshev number $C_v(m)$ is the smallest integer, such that if $x \geq C_v(m)$, then $\vartheta(x) - \vartheta(x/v) \geq m \ln x$, where $\vartheta(x) = \sum_{p \leq x} \ln p$ is the Chebyshev function.

Since $\frac{\vartheta(x) - \vartheta(x/v)}{\ln x}$ can enlarge on 1 only when x is prime, then all v -Chebyshev numbers $C_v(m)$ are primes.

Proposition 5. We have

$$(6) \quad R_v(m) \leq C_v(m).$$

Proof. Let $x \geq C_v(m)$. Then we have

$$(7) \quad m \leq \frac{\vartheta(x) - \vartheta(x/v)}{\ln x} = \sum_{\frac{x}{v} < p \leq x} \frac{\ln p}{\ln x} \leq \sum_{\frac{x}{v} < p \leq x} 1 = \pi(x) - \pi(x/v).$$

Thus, if $x \geq C_v(m)$, then *always* $\pi(x) - \pi(x/v) \geq m$. By the Definition 3, this means that $R_v(m) \leq C_v(m)$. \square

Now we give an upper estimates for $C_v(m)$ and $R_v(m)$.

Proposition 6. *Let $x = x_v(m) \geq 2$ be any number for which*

$$(8) \quad \frac{x}{\ln x} \left(1 - \frac{1300}{\ln^4 x}\right) \geq \frac{vm}{v-1}.$$

Then

$$(9) \quad R_v(m) \leq C_v(m) \leq x_v(m).$$

Proof. We use the following inequality of Dusart [3] (see his Theorem 5.2):

$$|\vartheta(x) - x| \leq \frac{1300x}{\ln^4 x}, \quad x \geq 2.$$

Thus we have

$$\begin{aligned} \vartheta(x) - \vartheta(x/v) &\geq x \left(1 - \frac{1}{v} - 1300 \left(\frac{1}{\ln^4 x} - \frac{1}{v \ln^4 \frac{x}{v}}\right)\right) \\ &\geq x \left(1 - \frac{1}{v}\right) \left(1 - \frac{1300}{\ln^4 x}\right). \end{aligned}$$

If now

$$x \left(1 - \frac{1}{v}\right) \left(1 - \frac{1300}{\ln^4 x}\right) \geq m \ln x, \quad x \geq x_v(m),$$

then

$$\vartheta(x) - \vartheta(x/v) \geq m \ln x, \quad x \geq x_v(m)$$

and, by the Definition 4, $C_v(m) \leq x_v(m)$. So, according to (6), we conclude that $R_v(m) \leq x_v(m)$. \square

Remark 7. In fact, in Theorem 5.2 [3] Dusart gives several inequalities of the form

$$|\vartheta(x) - x| \leq \frac{ax}{\ln^b x}, \quad x \geq x_0(a, b).$$

In the proof we used the maximal value $b = 4$. However, with the computer point of view, the values $a = 1300$, $b = 4$ from Dusart's theorem not always are the best. The analysis for $x \geq 25$ shows that the condition

$$x \left(1 - \frac{1}{v}\right) \left(1 - \frac{ax}{\ln^b x}\right) \geq m \ln x$$

is the weakest and thus satisfies for the smallest $x_v(x) = x_v(a, b)$, if to use the following values of a and b from Dusart's theorem:

$$\begin{aligned} a &= 3.965, \quad b = 2 \text{ for } x \text{ in range } (25, 7 \cdot 10^7]; \\ a &= 1300, \quad b = 4 \text{ for } x \text{ in range } (7 \cdot 10^7, 10^9]; \\ a &= 0.001, \quad b = 1 \text{ for } x \text{ in range } (10^9, 8 \cdot 10^9]; \\ a &= 0.78, \quad b = 3 \text{ for } x \text{ in range } (8 \cdot 10^9, 7 \cdot 10^{33}]; \\ a &= 1300, \quad b = 4 \text{ for } x > 7 \cdot 10^{33}. \end{aligned}$$

Proposition 6 gives the terms of sequences $\{C_v(m)\}$, $\{R_v(m)\}$ for every $v > 1$, $m \geq 1$. In particular, if $k = 1$ we find $\{C_2(m)\}$:

$$\begin{aligned} &11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, 179, 223, \\ &229, 233, 239, 241, 263, 269, 281, 307, 311, 347, 349, 367, 373, 401, 409, \\ (10) \quad &419, 431, 433, 443, \dots \end{aligned}$$

This sequence requires a separate comment. We observe that up to $C_2(100) = 1489$ only two terms of this sequence ($C_2(17) = 223$ and $C_2(36) = 443$) are not Ramanujan numbers, and the sequence is missing only the following Ramanujan numbers: 181, 227, 439, 491, 1283, 1301 and no others up to 1489. The latter observation shows how much the ratio $\frac{\vartheta(x)}{\ln x}$ exactly approximates $\pi(x)$.

Further, for $v = \frac{k+1}{k}$, we find the following sequences:
for $k = 2$, $\{C_v(m)\}$,

$$(11) \quad 13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \dots;$$

for $k = 2$, $\{R_v(m)\}$,

$$(12) \quad 2, 13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \dots;$$

for $k = 3$, $\{C_v(m)\}$,

$$(13) \quad 29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \dots;$$

for $k = 3$, $\{R_v(m)\}$,

$$(14) \quad 11, 29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \dots;$$

for $k = 5$, $\{C_v(m)\}$,

$$(15) \quad 59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \dots;$$

for $k = 5$, $\{R_v(m)\}$,

$$(16) \quad 29, 59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \dots;$$

for $k = 9$, $\{C_v(m)\}$,

$$(17) \quad 223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \dots ;$$

for $k = 9$, $\{R_v(m)\}$,

$$(18) \quad 127, 223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \dots ;$$

for $k = 14$, $\{C_v(m)\}$,

$$(19) \quad 307, 347, 563, 569, 733, 821, 1427, 1429, 1433, 1439, 1447, 1481, \dots ;$$

for $k = 14$, $\{R_v(m)\}$,

$$(20) \quad 127, 307, 347, 563, 569, 733, 1423, 1427, 1429, 1433, 1439, 1447, \dots .$$

4. ESTIMATES OF TYPE (3)

Proposition 8. *We have*

$$(21) \quad C_2(m-1) \leq p_{3m}, \quad m \geq 2;$$

$$(22) \quad R_{\frac{3}{2}}(m) \leq p_{4m}, \quad m \geq 1; \quad C_{\frac{3}{2}}(m-1) \leq p_{4m}, \quad m \geq 2;$$

$$(23) \quad R_{\frac{4}{3}}(m) \leq p_{6m}, \quad m \geq 1; \quad C_{\frac{4}{3}}(m-1) \leq p_{6m}, \quad m \geq 2;$$

$$(24) \quad R_{\frac{6}{5}}(m) \leq p_{11m}, \quad m \geq 1; \quad C_{\frac{6}{5}}(m-1) \leq p_{11m}, \quad m \geq 2;$$

$$(25) \quad R_{\frac{10}{9}}(m) \leq p_{31m}, \quad m \geq 1; \quad C_{\frac{10}{9}}(m-1) \leq p_{31m}, \quad m \geq 2;$$

$$(26) \quad R_{\frac{15}{14}}(m) \leq p_{32m}, \quad m \geq 1; \quad C_{\frac{15}{14}}(m-1) \leq p_{32m}, \quad m \geq 2.$$

Proof. Firstly let us find some values of $m_0 = m_0(k)$, such that, at least, for $m \geq m_0$ all formulas (21)-(26) hold. According to (8)-(9), it is sufficient to show that, for $m \geq m_0$, we can take p_{tm} , where $t = 3, 4, 6, 11, 31, 32$ for formulas (21)-(26) respectively, in the capacity of $x_v(m)$. As we noted in Remark 7, in order to get possibly smaller values of m_0 , we use, instead of (8), the estimate

$$(27) \quad \frac{x}{\ln x} \left(1 - \frac{3.965}{\ln^2 x} \right) \geq \frac{vm}{v-1}.$$

In order to get $x = p_{mt}$ satisfying this inequality, note that [11]

$$p_n \geq n \ln n.$$

Therefore, it is sufficient to consider p_{mt} satisfying the inequality

$$\ln p_{tm} \leq \left(1 - \frac{1}{v} \right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))} \right).$$

On the other hand, for $n \geq 2$, (see (4.2) in [3])

$$\ln p_n \leq \ln n + \ln \ln n + 1.$$

Thus it is sufficient to choose m so large that the following inequality holds

$$\ln(tm) + \ln \ln(tm) + 1 \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right),$$

or, since $1 - \frac{1}{v} = \frac{1}{k+1}$, that

$$(28) \quad \frac{\ln(tm) + \ln \ln(tm) + 1}{\ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right)} \leq \frac{t}{k+1}.$$

Let, e.g., $k = 1$, $t = 3$. We can choose $m_0 = 350$. Then the left hand side of (28) equals $1.4976... < 1.5$. This means that at least, for $m \geq 350$, the estimate (3) and, for $m \geq 351$, the estimate (21) are valid. Using a computer verification for $m \leq 350$, we obtain both of these estimates. Note that another short proof of (3) was obtained in [12] (see there Remark 32). Other estimates of the proposition are proved in the same way. \square

5. ESTIMATES AND FORMULAS FOR $N_k(m)$

Proposition 9.

$$(29) \quad N_k(1) = 2, \quad k = 2, 3, 5, 9, 14.$$

For $m \geq 2$,

$$(30) \quad N_k(m) \leq \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil;$$

besides, if $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$, then

$$(31) \quad N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k}{k+1}$$

and, if $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$, then

$$(32) \quad N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k - 1}{k+1}.$$

Proof. If $m \geq 2$, formally the condition $x = (k+1)n \geq (k+1)N_k(m)$ is not stronger than the condition $x \geq R_{\frac{k+1}{k}}(m)$, since the first one is valid only for x multiple of $k+1$. Therefore, for $m \geq 2$, (30) holds. It allows to calculate the terms of sequence $\{N_k(m)\}$ for every $k > 1$, $m \geq 2$. Since $N_k(1) \leq N_k(2)$, then, having $N_k(2)$, we also can prove (29), using direct calculations. Now

let $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$. Note that, for $y = (R_{\frac{k+1}{k}}(m) - 1)/(k+1)$ the interval

$$(33) \quad (ky, (k+1)y) = \left(\frac{k}{k+1} \left(R_{\frac{k+1}{k}}(m) - 1 \right), R_{\frac{k+1}{k}}(m) - 1 \right)$$

cannot contain more than $m - 1$ primes. Indeed, it is an interval of type $(\frac{k}{k+1}x, x)$ for integer x and the following such interval is

$$\left(\frac{k}{k+1} \left(R_{\frac{k+1}{k}}(m) \right), R_{\frac{k+1}{k}}(m) \right).$$

By the definition, $R_{\frac{k+1}{k}}(m)$ is the *smallest* number such that if $x \geq R_{\frac{k+1}{k}}(m)$, then $\{(\frac{k}{k+1}x, x)\}$ contains $\geq m$ primes. Therefore, the supposition that already interval (33) contains $\geq m$ primes contradicts the minimality of $R_{\frac{k+1}{k}}(m)$. Since the following interval of type $(ky, (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 1)$ is

$$\left(\frac{k}{k+1} (R_{\frac{k+1}{k}}(m) + k), R_{\frac{k+1}{k}}(m) + k \right),$$

then (31) follows.

Finally, let $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$. Again show that, for $y = (R_{\frac{k+1}{k}}(m) - 2)/(k+1)$ the interval

$$(34) \quad (ky, (k+1)y) = \left(\frac{k}{k+1} (R_{\frac{k+1}{k}}(m) - 2), R_{\frac{k+1}{k}}(m) - 2 \right)$$

cannot contain more than $m - 1$ primes. Indeed, comparing interval (34) with interval (33), we see that they contain the same integers except for $R_{\frac{k+1}{k}}(m) - 2$ which is multiple of $k+1$. Therefore, they contain the same number of primes and this number does not exceed $m - 1$. Again, since the following interval of type $(ky, (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 2)$ is

$$\left(\frac{k}{k+1} (R_{\frac{k+1}{k}}(m) + k - 1), R_{\frac{k+1}{k}}(m) + k - 1 \right),$$

then (32) follows. \square

Remark 10. Obviously formulas (30)-(32) are valid for not only for the considered values of k , but for arbitrary $k \geq 1$.

As a corollary from (29), (31)-(32), we obtain the following formula in case $k = 2$.

Proposition 11.

$$(35) \quad N_2(m) = \begin{cases} 2, & \text{if } m = 1, \\ \left\lceil \frac{R_{\frac{3}{2}}(m)}{3} \right\rceil, & \text{if } m \geq 2. \end{cases}$$

Formula (35) shows that the case $k = 2$ over its regularity not concedes to a classic case $k = 1$. Note that, if $k \geq 3$ and $R_{\frac{k+1}{k}}(m) \equiv j \pmod{k+1}$, $3 \leq j \leq k$, then, generally speaking, (30) is not an equality. Evidently, $N_k(m) \geq N_k(m-1)$ and it is interesting that the equality is attainable (see below sequences (37)-(40)).

Example 12. Let $k = 3$, $m = 2$. Then $v = \frac{4}{3}$ and, by (14), $R_{\frac{4}{3}}(2) = 29 \equiv 1 \pmod{4}$. Therefore, by (31), $N_3(2) = \frac{29+3}{4} = 8$. Indeed, interval $(3 \cdot 7, 4 \cdot 7)$ already contains only prime 23.

Example 13. Let $k = 3$, $m = 3$. Then, by (14), $R_{\frac{4}{3}}(3) = 59 \equiv 3 \pmod{4}$. Here $N_3(3) = 11$ which is essentially less than $\left\lceil R_{\frac{4}{3}}(3)/4 \right\rceil = 15$. Indeed, each interval

$(3 \cdot 15, 4 \cdot 15)$, $(3 \cdot 14, 4 \cdot 14)$, $(3 \cdot 13, 4 \cdot 13)$, $(3 \cdot 12, 4 \cdot 12)$, $(3 \cdot 11, 4 \cdot 11)$ contains more than 2 primes and only interval $(3 \cdot 10, 4 \cdot 10)$ contains only 2 primes.

In any case, Proposition 9 allows to calculate terms of sequence $\{N_k(m)\}$ for every considered values of k . So, we obtain the following few terms of $\{N_k(m)\}$:

for $k = 2$,

$$(36) \quad 2, 5, 13, 14, 23, 25, 33, 43, 46, 58, 60, 61, 71, 77, 80, 88, 103, 104, \dots ;$$

for $k = 3$,

$$(37) \quad 2, 8, 11, 17, 26, 38, 40, 41, 48, 57, 68, 68, 70, 87, 96, 100, 108, 109, \dots ;$$

for $k = 5$,

$$(38) \quad 2, 7, 17, 24, 25, 38, 41, 58, 59, 64, 65, 73, 95, 97, 103, 106, 107, 108, \dots ;$$

for $k = 9$,

$$(39) \quad 2, 14, 23, 23, 34, 36, 57, 58, 60, 60, 77, 86, 100, 100, 102, 123, 149, \dots ;$$

for $k = 14$,

$$(40) \quad 2, 11, 24, 37, 38, 39, 50, 96, 96, 96, 96, 97, 97, 125, 125, 132, 178, 178, \dots .$$

Remark 14. If, as in [1], [6], instead of intervals $(kn, (k+1)n)$, to consider intervals $[kn, (k+1)n]$, then sequences (5), (36)-(38) would begin with 1.

6. METHOD OF SMALL INTERVALS

If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $(x, (1+\frac{1}{\Delta})x]$ contains a prime, then we can calculate a bounded number of the first terms of sequences (5) and (36)-(40). Indeed, put $x_1 = kn$, such that $n \geq \frac{x_0}{k}$. Then $(k+1)n = \frac{k+1}{k}x_1$ and, if $1 + \frac{1}{\Delta} < \frac{k+1}{k}$, i.e., $\Delta > k$, then

$$\left(x_1, \left(1 + \frac{1}{\Delta}\right)x_1\right] \subset (kn, (k+1)n).$$

Thus, if $n \geq \frac{x_0}{k}$, then the interval $(kn, (k+1)n)$ contains a prime, and, using method of finite descent, we can find $N_k(1)$. Further, put $x_2 = (1 + \frac{1}{\Delta})x_1$. Then interval $(x_2, (1 + \frac{1}{\Delta})x_2]$ also contains a prime. Thus the union

$$\left(x_1, \left(1 + \frac{1}{\Delta}\right)x_1\right] \cup \left(x_2, \left(1 + \frac{1}{\Delta}\right)x_2\right] = \left(x_1, \left(1 + \frac{1}{\Delta}\right)^2 x_1\right]$$

contains at least two primes. This means that if $(1 + \frac{1}{\Delta})^2 x_1 < (k+1)n$ or $(1 + \frac{1}{\Delta})^2 < 1 + \frac{1}{k}$, then

$$\left(x_1, \left(1 + \frac{1}{\Delta}\right)^2 x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least two primes; again, using method of finite descent, we can find $N_k(2)$, etc., if $(1 + \frac{1}{\Delta})^m < 1 + \frac{1}{k}$, then

$$\left(x_1, \left(1 + \frac{1}{\Delta}\right)^m x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least m primes and we can find $N_k(m)$. In this way, we can find $N_k(m)$ for $m < \frac{\ln(1+\frac{1}{k})}{\ln(1+\frac{1}{\Delta})}$. In 2002, Ramaré and Saouter [9] proved that interval $(x(1 - 28314000^{-1}), x)$ always contains a prime if $x > 10726905041$, or, equivalently, interval $(x, (1 + 28313999^{-1})x)$ contains a prime if $x > 10726905419$. This means that, e.g., we can find $N_{14}(m)$ for $m \leq 1954471$. Unfortunately, this method cannot give the exact estimates and formulas for $N_k(m)$ as (30)-(32).

We can also to consider a more general application of of this method. Consider a fixed infinite set P of primes which we call P -primes. Furthermore, consider the following generalization of v -Ramanujan numbers.

Definition 15. For $v > 1$, a (v, P) -Ramanujan number $(R_v^{(P)}(m))$, is the smallest integer such that if $x \geq R_v^{(P)}(m)$, then $\pi_P(x) - \pi_P(x/v) \geq m$, where $\pi_P(x)$ is the number of P -primes not exceeding x .

Note that every (v, P) -Ramanujan number is P -prime. If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $(x, (1 + \frac{1}{\Delta})x]$ contains a P -prime, then, using the above described algorithm we can calculate a bounded number of the first (v, P) -Ramanujan numbers. For example, let P be the set of primes $p \equiv 1 \pmod{3}$. From the result of Cullinan and Hajir [2] it follows, in particular, that for $x \geq 106706$, the interval $(x, 1.048x)$ contains a P -prime. Using the considered algorithm, we can calculate the first 14 $(2, P)$ -Ramanujan numbers. They are

$$(41) \quad 7, 31, 43, 67, 97, 103, 151, 163, 181, 223, 229, 271, 331, 337.$$

Analogously, if P is the set of primes $p \equiv 2 \pmod{3}$, then the sequence of $(2, P)$ -Ramanujan numbers begins

$$(42) \quad 11, 23, 47, 59, 83, 107, 131, 167, 227, 233, 239, 251, 263, 281, \dots;$$

if P is the set of primes $p \equiv 1 \pmod{4}$, then the sequence of $(2, P)$ -Ramanujan numbers begins

$$(43) \quad 13, 37, 41, 89, 97, 109, 149, 229, 233, 241, 257, 277, 281, 317, \dots;$$

and, if P is the set of primes $p \equiv 3 \pmod{4}$, then the sequence of $(2, P)$ -Ramanujan numbers begins

$$(44) \quad 7, 23, 47, 67, 71, 103, 127, 167, 179, 191, 223, 227, 263, 307, \dots.$$

Denote by $N_k^{(P)}(m)$ the smallest number such that, for $n \geq N_k^{(P)}(m)$, the interval $(kn, (k+1)n)$ contains at least m P -primes. It is easy to see that formulas (30)-(32) hold for $N_k^{(P)}(m)$ and $R_{\frac{k+1}{k}}^{(P)}(m)$. In particular, in cases $k = 1, 2$ we have the formulas

$$(45) \quad N_1^{(P)}(m) = \frac{R_2^{(P)}(m) + 1}{2}, \quad N_2^{(P)}(m) = \left\lceil \frac{R_{\frac{3}{2}}^{(P)}(m)}{3} \right\rceil.$$

Therefore, the following sequences for $N_1^{(P)}(m)$ for the considered cases of set P correspond to sequences (41)-(44) respectively:

$$(46) \quad 4, 16, 22, 34, 49, 52, 76, 82, 91, 112, 115, 136, 166, 169, \dots;$$

$$(47) \quad 6, 12, 24, 30, 42, 54, 66, 84, 114, 117, 120, 126, 132, 141, \dots;$$

$$(48) \quad 7, 19, 21, 45, 49, 55, 75, 115, 117, 121, 129, 139, 141, 159, \dots;$$

$$(49) \quad 4, 12, 24, 34, 36, 52, 64, 84, 90, 96, 112, 114, 132, 154, \dots.$$

7. PROOF OF THEOREM 1

For $k \geq 1$, denote by $a(k)$ the least integer $n > 1$ for which the interval $(kn, (k+1)n)$ contains no prime; in the case, when such n does not exist, we put $a(k) = 0$. Taking into account (21), note that $a(k) = 0$ for $k = 1, 2, 3, 5, 9, 14, \dots$. Consider sequence $\{a(k)\}$. Its first few terms are (A218831 in [13])

$$(50) \quad 0, 0, 0, 2, 0, 4, 2, 3, 0, 2, 3, 2, 2, 0, 6, 2, 2, 3, 2, 6, 3, 2, 4, 2, 2, 7, 2, 2, 4, 3, \dots$$

Calculations of $a(k)$ in the range $\{15, \dots, 5 \times 10^7\}$ lead to values of $a(k)$ in the interval $[2, 16]$ which completes the proof. \square

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